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International Journal of Multiphase Flow 30 (2004) 1177–1192

International Journal of Multiphase Flow

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Effect of miscibility on the linear instability of two-fluid channel flow

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Received 14 August 2003; received in revised form 27 June 2004

Abstract

This study provides an extension into the finite-miscibility regime of theories for interfacial instabilities in two-fluid channel flow. When the two fluids are of high mass diffusivity, a new mode of instability, called here the overlap mode, appears when the critical layer of the dominant disturbance overlaps the viscosity-stratified layer. For poor diffusivity, the instability is qualitatively different: broadband and at low Reynolds number, beginning to resemble the Yih mode for immiscible interfaces. The results lend themselves to verification by experiment and direct numerical simulation.

Keywords: Instability of two-fluid flow; Miscibility; Viscosity-stratification

1. Introduction

We consider here the symmetric flow through a channel of two miscible fluids of equal densities but different viscosities separated by a mixed layer of viscosity-stratified fluid, as shown in Fig. 1. The stability analysis presented may be used for any level of miscibility and for any (non-zero) thickness of the mixed layer. The limiting case of immiscible fluids separated by a sharp interface has been studied extensively (e.g. Yih, 1955, 1967; Renardy, 1987; Joseph and Renardy, 1993;

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0301-9322/$ - see front matter
doi:10.1016/j.ijmultiphaseflow.2004.06.006

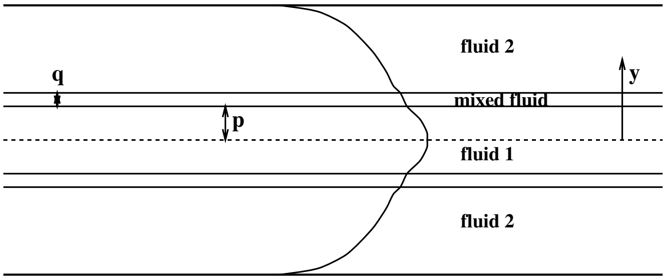

Fig. 1. Schematic of the flow.

Pinarbasi and Liakopolous, 1995; Laure et al., 1997; Hooper and Boyd, 1987; South and Hooper, 1999; Timoshin and Hooper, 2000), and the present work forms an extension of this body of work to miscible fluids. The other limiting case of stratified flow, which is equivalent here to a mixed layer extending across the entire channel, has again been studied by several authors (e.g. Wall and Wilson, 1996, 1997; Wazzan et al., 1970). Surprisingly, the stability behaviour of miscible two-fluid flow with a relatively thin mixed layer departs qualitatively from both interface-dominated flows and stratified flows. Several experiments have been carried out on miscible two-fluid flow (see e.g. Scoffoni et al., 2001; Lajeunesse et al., 1999; Petitjeans and Maxworthy, 1996; Fernandez et al., 2002; Cao et al., 2003) in different geometries, and a variety of interesting instabilities are observed. Instabilities driven by viscosity-stratification at low miscibility are found to be qualitatively similar to those observed in immiscible fluids (Scoffoni et al., 2001; Lajeunesse et al., 1999; Petitjeans and Maxworthy, 1996), this will be seen below to be consistent with present findings. Motivated by the fact that miscible two-fluid shear flows are common in the chemical industry (the symmetric pipe flow of molten plastic and a dye, for example), it is the aim of this paper to analyse in detail the stability of this flow.

It has been demonstrated in a preliminary study by Ranganathan and Govindarajan (2001) that when the critical layer of the dominant disturbance overlaps the mixed (viscosity-stratified) layer, the flow can be significantly stabilized or destabilized, depending on whether the fluid in the center (fluid 1) is more or less viscous than that close to the wall (fluid 2). At relatively high Reynolds numbers, the critical layer is a thin layer lying around the critical point y_c, where the mean flow velocity $U(y_c)$ is equal to the phase speed c of the dominant disturbance. In shear flows, the production of disturbance kinetic energy is often maximum within or close to the critical layer. (It is for this reason that this layer dictates instability in the present study as well.) When fluid 2 is less viscous, it has been shown (Govindarajan et al., 2001, 2003) that the huge stabilization of a given mode is due to a large negative production of disturbance kinetic energy (transfer of kinetic energy from the disturbance to the mean) under conditions of overlap. These studies were for fluids of infinite diffusivity: it is the primary aim of the present analysis to study diffusion effects. It is shown here that at high diffusivities, and when fluid 2 is more viscous, the interaction between the critical layer and the viscosity gradient leads to a new (overlap or O) mode of instability, distinct from both the well-known inviscid (I) and Tollmien–Schlichting/Schubauer (TS) modes. The overlap instability occurs at much lower Reynolds numbers than the others (although, for the range of diffusivities under consideration, the Reynolds numbers are still $\gg 1$) and is a consequence of the transformation of a TS excitation by gradients in the basic viscosity. The nature of the instability is shown to change qualitatively as the diffusivity goes to zero, and the behaviour begins to resemble that of immiscible fluids. The spatial development of the flow due to the downstream growth of the mixed layer is neglected here, but will be considered separately elsewhere: this approximation is valid for Schmidt numbers greater than 1. We take the fluids to be Newtonian.

2. Formulation

The formulation is described here for the upper half of the channel. The basic viscosity, being directly related to the basic concentration profile, is given by $\bar{\mu}_1$ and $\bar{\mu}_2$ for fluids 1 and 2 respectively and varies monotonically between these values in the mixed layer. The stability results are quite insensitive to the exact nature of the viscosity function $\bar{\mu}_m$ used in the mixed layer—linear and cubic viscosity profiles give very similar results. This should be reassuring for the potential experimenter, since the instability is basically triggered by some smooth variation across a thin layer. Numerical accuracy is better when the first two derivatives of the viscosity are continuous at the edges of the mixed layer, and as a simple way of ensuring this a fifth order polynomial is prescribed (the results have been checked to be indistinguishable from those obtained using a hyperbolic tangent viscosity profile):

$$\bar{\mu}_m = 1 + (m-1)\xi^3[10 - 15\xi + 6\xi^2], \quad 0 \leqslant \xi \leqslant 1, \tag{1}$$

where the mixed layer coordinate is defined by

$$\xi \equiv (y-p)/q, \tag{2}$$

where y is the coordinate normal to the wall with its origin at the channel centerline, q is the thickness of the mixed layer, $0 \leqslant y \leqslant p$ is the extent of fluid 1, and the viscosity ratio is given by $m \equiv \bar{\mu}_2/\bar{\mu}_1$. In Eq. (1) and in the subsequent discussion, all viscosities are scaled by μ_1. All lengths and velocities are non-dimensionalised using the half-width H of the channel and the centerline velocity U_0 as scales respectively.

The basic flow is obtained by requiring the velocity and all relevant derivatives to be continuous at the edges of the mixed layer.

$$U = 1 - \frac{Gy^2}{2}, \quad y \leqslant p, \tag{3}$$

where G is a non-dimensional number proportional to the mean streamwise pressure gradient $[G \equiv H^2/\bar{\mu}_1 U_0(\mathrm{d}P/\mathrm{d}x)]$,

$$U = U(p) - G\int_p^y \frac{y}{\bar{\mu}_m}\,\mathrm{d}y, \quad p \leqslant y \leqslant p+q, \tag{4}$$

and

$$U = \frac{G}{2m}(1-y^2), \quad y \geqslant p+q. \tag{5}$$

In view of Yih's (1955) extension of Squire's theorem to viscosity-stratified flows, it is sufficient to consider two-dimensional disturbances, since they are the least stable. We consider the stability of the flow to linear perturbations in the velocity and viscosity profiles. Perturbations in viscosity are directly related to perturbations in concentration; within the mixed layer, for example,

$$\mu_{\text{tot}} = \bar{\mu} + \hat{\mu} = 1 + (m-1)f_2 = 1 + (m-1)[\bar{f}_2 + \hat{f}_2], \tag{6}$$

where f_2 is the fraction of fluid 2 at a given height, giving

$$\hat{\mu} = (m-1)\hat{f}_2. \tag{7}$$

The linear stability of this flow is governed by the "thermal" Orr–Sommerfeld equations (Wall and Wilson, 1996), which are derived from the 2D Navier–Stokes and scalar transport equations respectively by the standard procedure: the flow quantities (velocities and concentrations) are split into their respective means and perturbations. The perturbations are written in the normal mode form as

$$[\hat{\phi}(x,y,t), \hat{\mu}(x,y,t)] = [\phi(y), \mu(y)]\exp[\mathrm{i}\alpha(x-ct)], \tag{8}$$

where x is the streamwise coordinate and t is time. In the Navier–Stokes and scalar transport equations, all nonlinear terms in the perturbation are neglected, and the mean flow equations subtracted out, resulting in

$$\begin{aligned}\mathrm{i}\alpha[(\phi'' - \alpha^2\phi)(U-c) - U''\phi] = \frac{1}{R}[&\bar{\mu}\phi^{\text{iv}} + 2\bar{\mu}'\phi''' + (\bar{\mu}'' - 2\alpha^2\bar{\mu})\phi'' - 2\alpha^2\bar{\mu}'\phi' + (\alpha^2\bar{\mu}'' + \alpha^4\bar{\mu})\phi \\ &+ U'\mu'' + 2U''\mu' + (U''' + \alpha^2 U')\mu]\end{aligned} \tag{9}$$

and

$$i\alpha[(U-c)\mu - \bar{\mu}'\phi] = \frac{1}{Pe}(\mu'' - \alpha^2\mu) \tag{10}$$

with the boundary conditions

$$\phi(\pm 1) = \phi'(\pm 1) = \mu(\pm 1) = 0. \tag{11}$$

The Reynolds number and the Peclet number are defined respectively as $R \equiv \rho U_0 H/\bar{\mu}_1$ and $Pe \equiv U_0 H/\kappa$, and the primes denote differentiation with respect to y. The density is ρ and κ is the mass (or thermal, when appropriate) diffusivity.

3. Results

Eqs. (9) and (10) are solved using a Chebychev collocation spectral method. The grid is clustered in the mixed layer using the stretching function

$$y_j = \frac{a}{\sinh(by_0)}[\sinh\{(y_C - y_0)b\} + \sinh(by_0)], \tag{12}$$

where y_C is a collocation point (81 collocation points are used here) and

$$y_0 = \left(\frac{0.5}{b}\right) \log\left[\frac{(1 + (e^{b} - 1)a)}{(1 + (e^{-b} - 1)a)}\right]. \tag{13}$$

The quantity a represents the location around which clustering is desired: it is chosen here to be the midpoint of the mixed layer. The factor b determines the degree of clustering, a value of between 4 and 8 gives eigenvalues accurate up to five decimal places. The thickness of the viscosity-stratified layer was fixed at $q = 0.1$ for most of the study, but the results are not qualitatively different for different q in the range $0.05 \leqslant q \leqslant 0.3$.

The most important parameter in this flow is the location p (distance from the axis) of the viscosity-stratified layer, which is determined by the relative mass fluxes of the two fluids. Stability boundaries for a viscosity ratio of $m = 1.20$ and $Pe = 0$ are shown for different p in Fig. 2. Here, R_{av} is the Reynolds number based on the spatially averaged viscosity across the channel. For small p, there is only one kind of instability in the flow, namely the TS mode, as in the flow of a single fluid. At $p \geqslant 0.25$, the velocity profile is inflexional, and a second instability appears at shorter wavelengths: it is termed here the 'I' or inviscid mode. At $p = 0.604$ (not shown), a new mode, disjoint from the TS and the I becomes unstable at low Reynolds numbers. Since the critical layer of this new mode extends into the mixed layer at this p, it is designated the overlap, or 'O' mode; the mechanism for its occurrence is discussed in the following section. At $p = 0.65$, all three instabilities may be seen in Fig. 2(c) to be distinct and occupying sizeable regions of the $\alpha - R$ plane. At $p = 0.6565$ the stability loop contains a bifurcation point; beyond this p, the O and I modes coalesce (Fig. 2(d) and (e)). The joint mode then goes through another coalescence with the TS mode through another singularity and all three instability regimes are

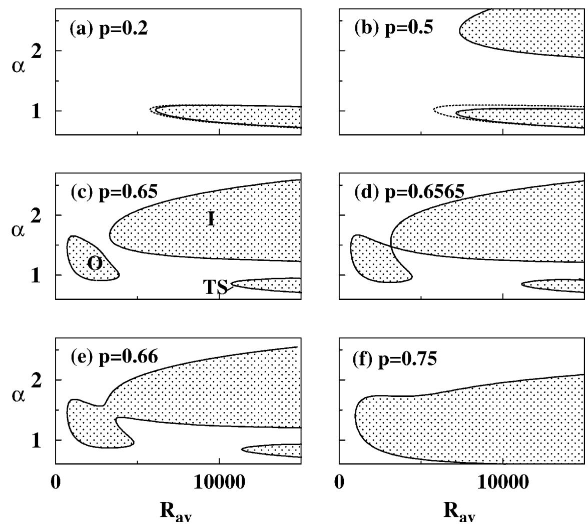

Fig. 2. Stability boundaries for $m = 1.20$, $Pe = 0$. Shaded regions: unstable. As the location of the mixed layer moves away from the center, the inviscid (I) mode and then the overlap (O) mode become dominant. A merger of these modes is seen for higher values of the mixed-layer location, p. The linear stability boundary for a single fluid is shown by the dotted lines in (a) and (b).

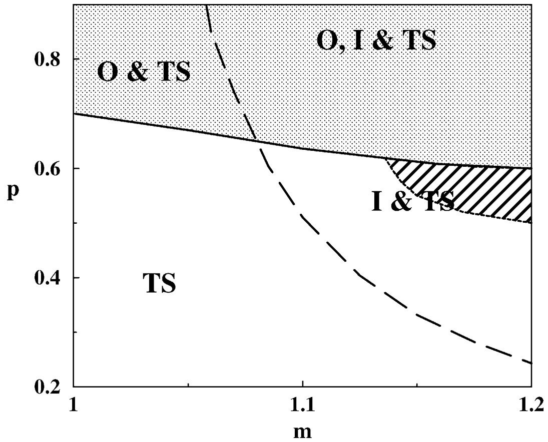

Fig. 3. Schematic of the regimes of existence and dominance of instabilities due to the O and I modes. Unstable TS modes exist everywhere in the regime. The O mode exists above the solid line and is dominant mode wherever it occurs (in the shaded region above the solid line), whereas the I mode exists to the right of the dashed line and is dominant over the TS only in the small shaded region shown.

indistinguishable (Fig. 2(f)) at high p. A similar coalescence of the TS with an interfacial mode has recently been observed in two-layer boundary layer flow by Timoshin and Hooper (2000).

The regimes of existence of the O and I modes over a range of viscosity ratios are shown in Fig. (3). The O mode occurs above the solid line in the figure, while the I mode exists everywhere to the right of the dashed line. Instabilities of the Tollmien–Schlichting (Schubauer) type exist throughout the regime shown. The question is, which mode is dominant, i.e., becomes unstable at lower Reynolds numbers than the others? The TS mode is dominant in the unshaded region in the figure. The O mode always becomes unstable at a lower Reynolds number than the others and is therefore dominant in the large shaded region in the figure, i.e., whenever the mixed layer lies closer to the wall than the solid line shown. When the mixed layer is well-separated from the critical layer, i.e., when p is less than the value on the solid line and when the velocity profile is not inflexional (to the left of the dashed line), only the TS mode exists. To the right of the dashed line, the velocity profile is inflexional. The I mode is however dominant over the TS only in a portion of this region represented by the small shaded area in Fig. 3.

We now examine the effects of reduced diffusivity; beginning with the case when the fluid viscosities differ by only 5%. Fig. 4 shows the case when fluid 2 is more viscous ($m = 1.05$). At high diffusivity levels, i.e., when the Schmidt number $Sc \sim 10$ or less ($Sc \equiv Pe/R$), the TS and overlap mode are merged. Given that a single fluid ($m = 1$) would become unstable only at $R = 5772$, it is remarkable how destabilizing a small difference in viscosity can be. The regimes of instability of the TS and O modes are coalesced at these Schmidt numbers. At intermediate levels of diffusivity ($Sc = 1000$ in the figure), the O mode is seen to be unstable in a domain distinct from that of the TS. For $Sc = 10^5$, i.e., when the fluids diffuse extremely slowly into each other, the behaviour is seen to be qualitatively different. The flow is unstable at very low Reynolds number and over a wide range of wavenumbers, resembling the well-known Yih mode of instability for immiscible interfaces. The opposite effect, namely, significant stabilization, is seen when the viscosity ratio is 0.95 (Fig. 5).

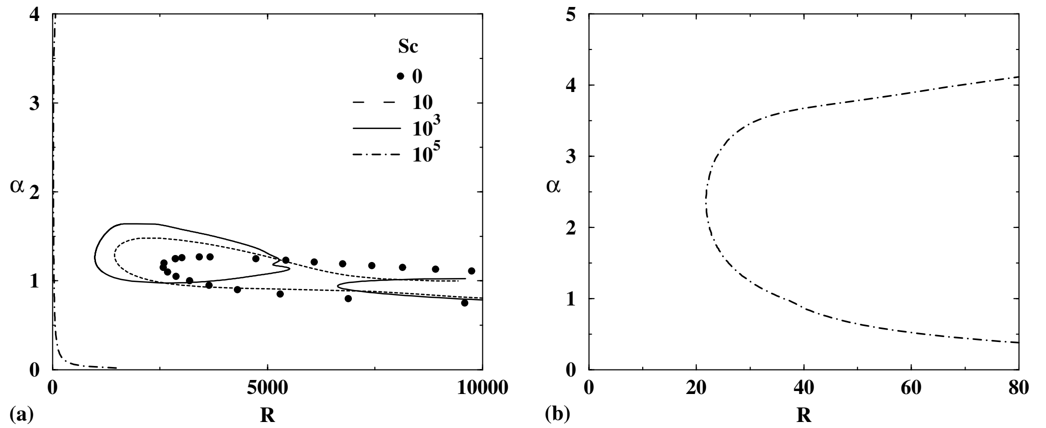

Fig. 4. Dependence of stability on diffusivity, $m = 1.05$, $p = 0.75$. Figure (b) is the low-Reynolds number portion of (a) magnified to show the stability boundary at $Sc = 10^5$. The phase speed of the dominant disturbance at the critical Reynolds number is 0.264 for $Sc = 0$, 0.367 for $Sc = 10$, 0.374 and 0.255 for $Sc = 10^3$ respectively for the overlap and the TS modes and 0.398 for $Sc = 10^5$.

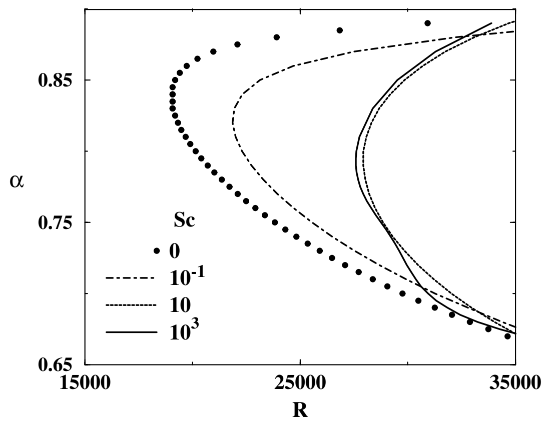

Fig. 5. Dependence of stability on diffusivity, $m = 0.95$, $p = 0.85$. The phase speeds in the order of increasing Schmidt number at the respective critical Reynolds numbers are 0.264, 0.148, 0.154 and 0.149 respectively.

For the case when the viscosity-stratified layer and the critical layer are well-separated, and the dominant instability is TS, the diffusivity has little effect on the stability: the region of instability does not change much (Fig. 6) when the Schmidt number is increased from 0 to 1000. This is consistent with the findings of Wall and Wilson (1996) who have shown that the stability of a stratified flow is insensitive to the Peclet number. The overlap mode, on the other hand, is affected significantly by Sc, as seen in Fig. 7. The degree of destabilization by the overlap mode goes up as the diffusivity decreases. The effect of diffusivity on the Reynolds number of the first instability, R_{inst}, is plotted in Fig. 8 for the situation when the mixed layer and critical layer are well separated, and in Fig. 9 when the two layers overlap. In the former, the diffusivity is seen to have

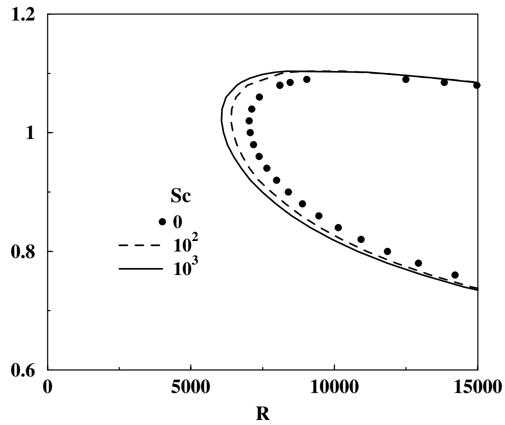

Fig. 6. Dependence of the TS mode on diffusivity, $m = 1.20$, $p = 0.20$. The phase speeds at the critical Reynolds number for $Sc = 100$ and 1000 are 0.265 and 0.269 respectively.

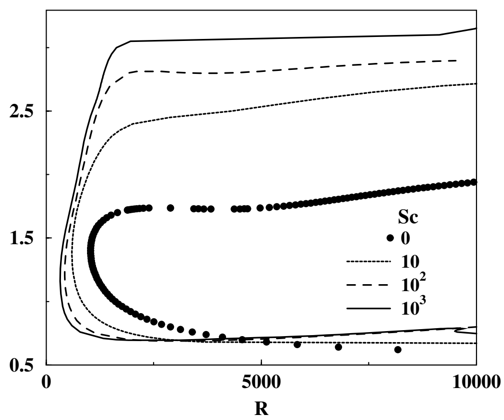

Fig. 7. Dependence of the overlap mode on diffusivity, $m = 1.20$, $p = 0.75$. The phase speeds at the critical Reynolds number in the order of increasing Schmidt number are 0.264, 0.396, 0.365 and 0.357 respectively.

little effect on the stability, in agreement with the results of Wall and Wilson (1996). The overlap mode, on the other hand, is strongly dependent on the diffusivity at Schmidt number higher than 0.1. When the core fluid is less viscous, ($m > 1$), decreasing diffusivity is consistently destabilizing, while the response for $m < 1$ is not monotonic. At $m \leqslant 0.92$, the instability Reynolds number for very poorly miscible fluids is independent of the viscosity ratio. The effect on transition to turbulence of the linear instability Reynolds numbers being pushed to very high values is yet to be understood. All we can say at this stage is that the amplitude of the primary disturbance is determined to a large extent by its stability, and since the growth of three-dimensional secondary modes requires significant (though small) amplitudes of the primary, a suppression of the primary results in a suppression of the secondary.

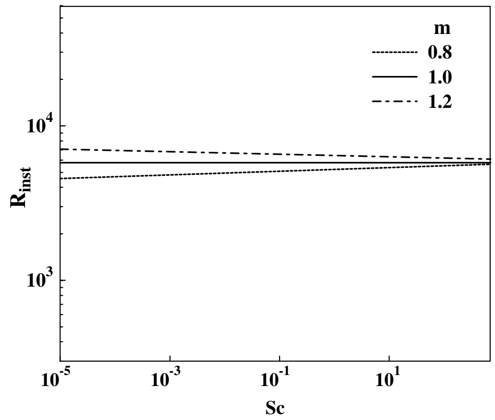

Fig. 8. Instability Reynolds number as a function of Schmidt number, $p = 0.2$, i.e., the critical and mixed layers are distinct from each other.

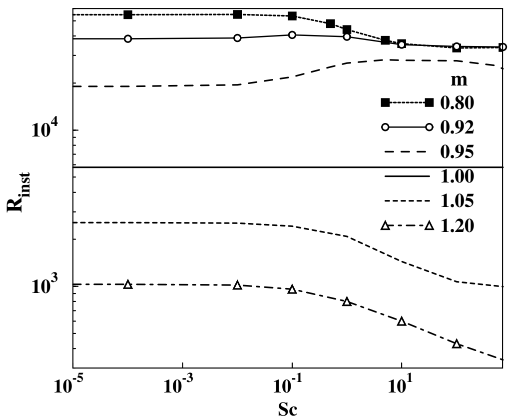

Fig. 9. Instability Reynolds number as a function of Schmidt number, overlap conditions.

4. What is happening in the critical layer?

We begin our discussion by showing (Fig. 10) where the critical layer lies with respect to the mixed-layer under well-separated and overlap conditions. Plotted in this figure is the location y_c of the critical point (where the phase speed of the disturbance equals the basic velocity) corresponding to Fig. 2(a) and (c). Consider first the case when the critical point is located far away from the mixed layer, i.e., when $p = 0.2$. Only the TS mode exists here. At $p = 0.65$, on the other hand, there are three trends in the critical point, corresponding to different modes of instability, as shown. The critical point y_c is surrounded by a critical layer where most of the disturbance kinetic energy is produced (Govindarajan et al., 2001), whose thickness is of the order ($R^{-1/3}$) as shown in Section 4. For Reynolds numbers of the order of 1000, the critical layer thickness is about 0.1. The balances in the production layer of the dominant TS mode in the case of $p = 0.65$ are very different due to the existence of the viscosity-stratification, and the O mode is created.

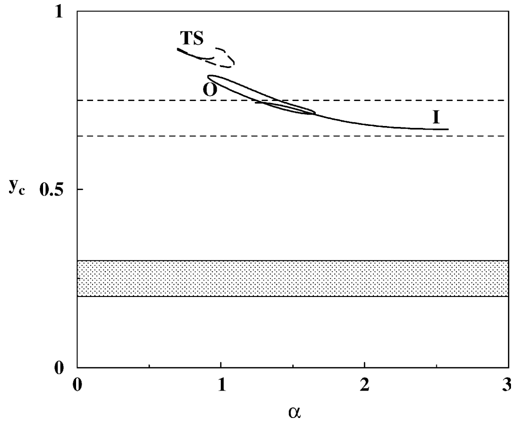

Fig. 10. Location of critical point y_c for each mode of instability, $m = 1.20$. Solid line: $p = 0.65$; dashed line: $p = 0.20$. The critical layer, as discussed in Section 4, extends to $\sim$O($R^{-1/3}$) around y_c. The mixed-layer for $p = 0.65$ lies between the pair of horizontal dashed lines while that for $p = 0.25$ is shown by the shaded region.

The results presented above indicate that interesting changes in the stability behaviour take place when the mixed layer and critical layer overlap. This is because disturbance kinetic energy is primarily produced within the critical layer and the overlap of a viscosity-stratified layer interferes with this production. At zero Peclet number (Govindarajan et al., 2001), for $m < 1$ (less viscous fluid near the wall), the overlap mechanism causes the total disturbance kinetic energy production to become large and negative, so that energy is being transferred at a rapid rate from the disturbance to the mean at a Reynolds number which would have been neutral if the stratified layer had been placed elsewhere in the channel. We now return to the stability Eqs. (9) and (10) to isolate the terms responsible for these changes. In order to do this, we perform an asymptotic scaling analysis in the critical layer similar to that described in Govindarajan and Narasimha (1999) (a detailed derivation of critical layer equations for boundary-layer flow under the parallel flow assumption is available in Lin, 1945a,b, 1946). The present problem contains three length scales in the critical layer: the typical thickness ϵ of momentum diffusion, the length scale δ of mass diffusion, and the mixed layer thickness, q. If the two fluids are perfectly immiscible, Sc is infinite, $q = 0$ and Eq. (10) reduces to the kinematic condition for the interface. This case has been studied by several authors, beginning with Yih (1967). Here, we consider $Sc < \infty$: the right hand side of Eq. (10) is significant (gradients in the disturbance viscosity are large) in some neighborhood near the mixed layer. The variables

$$\eta \equiv \frac{y - y_c}{\epsilon} \quad \text{and} \quad \zeta \equiv \frac{y - y_c}{\delta}, \tag{14}$$

defined in the direction normal to the wall are of O(1) within the layers of momentum and mass diffusion respectively. We expand the disturbance eigenfunctions in the critical layer in powers of the appropriate small parameter as

$$\phi(y) = \sum_k \epsilon^k \phi_k(\eta) \quad \text{and} \quad \mu(y) = \sum_k \delta^k \mu_k(\zeta), \quad k = 0, 1, 2, \ldots. \tag{15}$$

The mean flow velocity may be expanded about the critical point in a Taylor series:

$$U(y) - c = (y - y_c)U'_c + \frac{(y - y_c)^2}{2!}U''_c + \cdots. \tag{16}$$

We are interested in what happens under overlap conditions, i.e., the situation where all the action of mass and momentum diffusion is taking place in the same neighborhood. The normal derivatives of the mean viscosity under these conditions may be obtained as follows from Eqs. (1) and (2):

$$\bar{\mu}'(y) = \frac{30(m-1)\xi^2}{q}(1-\xi)^2 \equiv \frac{g_1(\xi)}{q}, \tag{17}$$

and

$$\bar{\mu}''(y) = \frac{60(m-1)\xi}{q^2}(1 - 3\xi + 2\xi^2) \equiv \frac{g_2(\xi)}{q^2}, \tag{18}$$

with ξ varying between 0 and 1 in the mixed layer. In this paper we are interested in the case $(m-1) \sim O(0.1)$ and the coefficients in (17) and (18) are of O(10); both g_1 and g_2 are therefore of O(1). Incidentally, if the fifth-order polynomial in (1) were to be replaced by any other realistic smooth function, a hyperbolic tangent, for example, we would still get derivatives of the same order as those above. The present choice of viscosity profile thus has no bearing on the ordering analysis.

We may write the derivatives of the mean velocity in the overlap layer from Eq. (4) as

$$U'(y) \sim O(1) \quad \text{and} \quad U''(y) = \frac{g_3(\xi)}{q} + O(1), \tag{19}$$

where $g_3 \equiv -Gpg_1/\bar{\mu}^2$ is of O(1).

Using the expansions (15), along with (16) to (19), Eqs. (9) and (10), may be reduced in the critical layer to

$$\mathrm{i}\alpha\eta[U'_c\phi''_0 + \chi g_{3c}\phi_0] = \frac{1}{\epsilon^3 R}\left[\bar{\mu}\phi_0^{\mathrm{iv}} - 2\chi g_{1c}\phi'''_0 + \chi^2 g_{2c}\phi''_0 + \frac{\epsilon^4}{\delta^2}U'_c\mu''_0\right] \tag{20}$$

and

$$\mathrm{i}\alpha Sc\left[\delta\zeta U'_c\mu_0 - \frac{g_{1c}}{q}\phi_0\right] = \frac{1}{R}\left[\frac{\mu''_0}{\delta^2}\right] \tag{21}$$

respectively. Here $\chi \equiv \epsilon/q$ is the ratio of the thickness of the critical layer to that of the mixed layer. For purposes of demonstration, the above equations have been written for $\bar{\mu} \sim O(1)$ and $\chi \sim O(1)$, but corresponding equations may of course be written for any $\bar{\mu}$ and χ. On comparing with the critical layer balance for a constant viscosity fluid (Lin, 1946) it is apparent that the critical-layer balance here is totally different (Eq. 21 would not exist for a single fluid, and equation 20 would contain only the first term on the left hand side and the first term on the right hand side), and that the stabilization/destabilization of the flow depends at the lowest order on the sign of $\bar{\mu}'$. On the other hand, for a simply stratified fluid, $\chi \ll 1$, and the behaviour does not change qualitatively from that of a single fluid. The lowest-order equations for a given relative magnitude of ϵ and δ may be derived from (20) and (21).

For infinitely miscible fluids, Eq. (10) (or, equivalently, (21)) gives the trivial solution $\mu_0 = 0$. This case has been studied in Govindarajan et al. (2001) where it was shown that the second term on the left hand side of Eq. (20) is the primary cause of the altered balances in the overlap layer. This term stems from the second derivative of the mean velocity, which is modified significantly by viscosity stratification. At low Schmidt numbers (high diffusivity), we expect that

$$\delta\mu_0 \ll \frac{\phi_0}{q}, \tag{22}$$

and Eq. (21) gives

$$-\frac{\mathrm{i}\alpha Sc g_{1c}}{q}\phi_0 = \frac{1}{R}\left[\frac{\mu_0''}{\delta^2}\right]. \tag{23}$$

Eq. (23) may be used to eliminate μ_0 from (20), giving

$$\bar{\mu}\phi_0^{\mathrm{iv}} - 2\chi g_{1c}\phi_0''' + [\chi^2 g_{2c} - \mathrm{i}\alpha U_c'\eta]\phi_0'' - \mathrm{i}\alpha\chi(\eta g_{3c} + g_{1c}Sc)\phi_0 = 0 \tag{24}$$

where we have defined

$$\epsilon \equiv R^{-1/3}. \tag{25}$$

For high diffusivity, the Schmidt number modifies the second inviscid term, i.e., reduced miscibility modulates the effect of the second derivative of the velocity profile. As the diffusivity decreases, the magnitude of the modulation increases. For $Sc \sim \mathrm{O}(1)$ or less, there is no qualitative change in the structure of the critical layer equation.

As the Schmidt number increases to higher values, however, slow diffusion effects will play an increasing role. From Eq. (20), we may estimate the Schmidt number Sc_p above which the stability behaviour may be expected to be dominated by poor miscibility at the 'interface'. At $Sc \geqslant Sc_p$, wall conditions are relatively unimportant, i.e., the first term on the right hand side is of higher order than the last term:

$$\bar{\mu}\phi_0^{\mathrm{iv}} \sim \mathrm{O}\left(\frac{\epsilon^4}{\delta^2}\mu_0''\right). \tag{26}$$

The thickness δ of the disturbance concentration critical layer may be estimated from Eq. (21). Since g_1 is proportional to $(m-1)$ by definition, we can write

$$\delta \sim \left[\frac{R_\mathrm{m}}{cPe}\frac{\mu_0''}{\phi_0}\right]^{1/2}, \tag{27}$$

where R_m is a mixed-layer Reynolds number given by

$$R_\mathrm{m} \equiv \frac{qc}{30(m-1)}. \tag{28}$$

Since $\phi_0^{\mathrm{iv}}(\eta) \sim \mathrm{O}[\phi_0(\eta)]$, we obtain the following condition under which poor miscibility effects will determine stability:

$$\frac{R_\mathrm{m}}{cPe}\frac{\mu_0}{\phi_0} \leqslant \mathrm{O}(\epsilon^5) \tag{29}$$

or

$$Sc_p \geqslant R_m R^{2/3}. \tag{30}$$

In the parameter range studied here, R_m/c varies from 0.5 to 2, and $R \sim 1000$, giving $Sc_p \sim 100$. The overlap instability is thus expected to be dominant when the mixed layer is caused by temperature differences or by the flow of two moderately miscible fluids (such as organic solvents). Since the Schmidt number decreases significantly with heating, interfacial instabilities are not expected to occur in the flow of hot fluids of comparable viscosities, here again the overlap instability would be dominant. For $Sc > Sc_p$, the present analysis provides an extension into the miscible regime of the Yih/Joseph type stability analysis for interfaces.

Since the overlap instability does not involve the introduction of an interface, it is to be expected that no new eigenmode is introduced into the system. In fact, the overlap instability arises

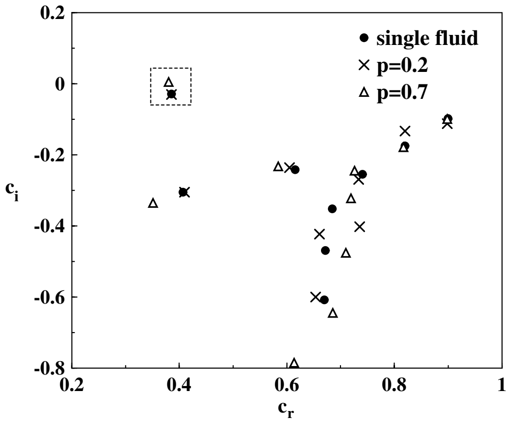

Fig. 11. Eigenvalue spectrum, $m = 1.20$, $R = 1000$, $\alpha = 1.25$. The least stable eigenmodes are indicated by a box.

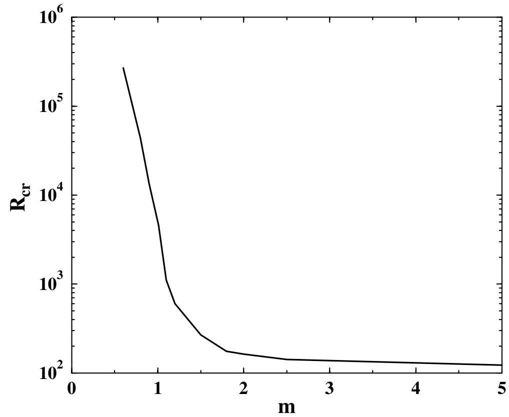

Fig. 12. The critical Reynolds number over a range of viscosity ratio. The values of p used are 0.6, 0.65, 0.7, 0.8, 0.85 and 0.89 for $m \geqslant 1.8$, $m = 1.5$, 1.1–1.2, 0.9–1.1, 0.8 and 0.6 respectively.

from the destabilization of an existing mode by a new balance in the critical layer. This is evident from Fig. 11, where eigenvalue spectra at $R = 1000$ for two locations of the mixed layer are shown: at $p = 0.2$ where the mixed layer and the critical layer (of the dominant disturbance) are well separated, and at $p = 0.7$, where they overlap. The spectra in the two cases look similar, but the flow in the latter case is definitely unstable while in the former case it is not. It is seen that the overlap eigenmode exists in single-fluid flow as well (it belongs to the 'A' family, according to the classification of Mack, 1976).

The viscosity ratios shown so far have been very close to 1, the main idea is to show that even a minor change in fluid properties can have an immense effect on stability. The critical Reynolds number for a wider range of viscosity ratio is shown in Fig. 12, it is seen that the major change is around $m = 1$. Typical values of p to trigger the overlap mechanism have been adopted here.

5. Conclusions

To summarise, we have studied the stability of the channel flow of two miscible fluids as a function of the miscibility. The present analysis provides an extension into the miscible regime of well-established theories for immiscible flows. At low values of diffusivity, the behaviour resembles that of the well-known interfacial mode. At higher levels of miscibility, the behaviour is qualitatively different, and over a wide range of diffusivities, a distinct new instability is evident. The new mode appears when the viscosity-stratified layer overlaps the critical layer of the dominant disturbance; it may be exploited to achieve a large stabilization/destabilization with minor differences in the viscosities of the two fluids. The predictions made here lend themselves to verification by experiment/direct numerical simulations.

The downstream growth of the mixed layer is inversely related to the Peclet number, as

$$\frac{\mathrm{d}q}{\mathrm{d}x} \sim \mathrm{O}(Pe)^{-1}\frac{1}{q}, \tag{31}$$

and, unless q is very small, neglecting this growth is a reasonable assumption at high Peclet numbers. A parallel (x-independent) viscosity-stratified layer would also exist in the flow of a single power-law fluid; the equivalent Schmidt number here is very low. Two fluids which are very rapidly diffusing, however, would give rise to a highly non-parallel mixed layer, even at high Reynolds numbers. Our analysis would be inaccurate in this case, and future studies including non-parallel effects are called for. Another line of investigation which needs to be pursued is the computation of algebraic/secondary growth of disturbances. The latter studies are sure to alter present estimates of the magnitude of stabilization/destabilization possible by the overlap mechanism, given that the transition to turbulence in the channel flow of a single fluid occurs at Reynolds numbers well below the instability Reynolds numbers of the primary (TS) mode. The actual flow at these Reynolds numbers may be taken to consist primarily of the mean flow plus (decaying) TS waves of finite but small amplitudes. It has been amply demonstrated (see e.g., Orszag and Kells, 1980; Reddy et al., 1998) that such a flow is unstable to algebraic/secondary modes, which are often three-dimensional. Our investigations on secondary instabilities (Govindarajan et al., 2003) in infinitely miscible two-fluid flow show that this mechanism has a large influence on the secondary instability as well. The reason for this is that the growth/decay of the secondary mode is dependent on the amplitude of the primary. Since the overlap mechanism can be used to alter hugely the amplitude of the primary disturbance, the fate of secondary modes is determined to a large extent. For example, when the outer fluid is less viscous by 10%, all TS waves decay to zero very quickly at typical Reynolds numbers of a few thousand, and three-dimensional secondary modes have little opportunity to grow. The existence of the overlap mode in other flows such as boundary layers and the possibility of its employment for flow control by means of heat or the introduction of a second fluid need to be explored.

Acknowledgment

The author thanks Prof. R. Narasimha, Prof. K. R. Sreenivasan and Dr. K. R. Sreenivas for their detailed comments. This work is funded by the Defence R&D Organisation, Government of India.

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